Delayed stochastic differential model for quiet standing

W. Yao, P. Yu, and C. Essex
Applied Mathematics Department, University of Western Ontario, London, Ontario, Canada N6A 5B7
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A physiological quiet standing model, described by a delayed differential equation, subject to a white noise perturbation, is proposed to study the postural control system of human beings. It has been found that the white noise destabilizes the equilibrium state, and inertia accelerates the destabilizing process, and that the position of a person is detected and processed by the person's nervous system with a delay. This paper focuses on the analysis of Hopf bifurcation and its stability in this context. Based on the analytical predictions confirmed by numerical simulations, it has been shown that the posture of a person is controlled in such a way that possible amplitude oscillations are minimized.

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I. INTRODUCTION

Standing quietly appears to the eye as a simple matter of being inert or at rest. However, precise measurements reveal that it actually involves complex dynamical motions, which are too small for the eye to see. Thus the simple act of standing is revealed to be a complex control problem, involving both physiological and psychological processes. Recent studies on quiet standing posture experimental data, measuring the trajectory of the center of pressure (COP) under the feet of quietly standing subjects, show that the behavior of young healthy subjects is like colored noise [1]. It has been found that the two-point autocorrelation function in the anteroposterior (front-to-back) direction, y, defined by $C(\Delta t)=\langle (y(t)-y(t-\Delta t))^2\rangle$ is proportional to $(\Delta t)^{2H}$, where the value of H defines three different dynamic regimes:

$$H\begin{cases} >0.5 & \text{for } 0\leqslant\Delta t\leqslant\Delta t_1 \\ <0.5 & \text{for } \Delta t_1<\Delta t\leqslant\Delta t_2 \\ \approx 0.0 & \text{for } \Delta t>\Delta t_2, \end{cases}$$

where Δt_1 and Δt_2 are observed values. One can use the values of H to determine whether or not the analyzed data are correlated. For example, a classical random walk corresponds to $H=0.5$, while $H>0.5$ and $H<0.5$ indicate positive and negative correlations, respectively [1]. From the control point of view, these observations suggest that quiet standing posture may thus be explained as follows: within some time (e.g., the subject's reaction time), small deviations grow, and are suppressed by a negative feedback on larger time scale.

Experimental models have been proposed to study quiet standing, and in particular, a similar shape of the function $C(\Delta t)$ has been obtained by adjusting parameters in the models proposed in Refs. [2,3]. Chow and Collins [2] used the transverse motion of an elastically pinned polymer to simulate human postural movements. This model can be described by a driven stochastic partial differential equation, when the polymer is perturbed by a noise forcing function. The noise is colored but its correlation is so simple that the differential equation can be treated analytically. It has been shown that the form of the autocorrelation function for the pinned polymer model is similar to the function $C(\Delta t)$, characterized by the three indicated regimes. Thus, the behavior of the highly evolved and complicated human postural control system can be simply, yet realistically, described by a linear system. This model is also suitable for studying a perturbed human postural control system [4]. The work of Ohira and Milton [3] has shown that human postural control can be described by an even simpler model—delayed random walks. Delay means a walker moves to the left or right, say, based on the walker's position at some previous times. This model is based on the natural delay in the nervous active process, and the behavior of the COP is like colored noise. The dynamical behavior of the resulting model also agrees with the characteristic (shape) of function $C(\Delta t)$.

Why does $C(\Delta t)$ exhibit such a shape? What advantages would a person get from the behavior implied by its shape? To our knowledge, no publications have addressed such questions. To answer them, we propose a simplified physiological model in this paper, utilizing a delayed stochastic nonlinear differential equation.

Section II presents the model. Section III considers Hopf bifurcation and its stability conditions. Discussions and conclusions are given in Sec. IV.

II. MODEL

A rough description of quiet standing process will help construct a model. Imagine a person standing still in a quiet stable state, and then consider some disturbances leading to some very complicated perturbations. Some natural disturbances would be the beating of a person's heart, the contraction of the stomach, etc. Such perturbations would cause the person's position to move away from the original stable equilibrium. When the initial displacement is too small for the person to feel, no adjustment to posture happens. However, as the displacement grows, due to the inertia and the loss of stability of the gravity center, the displacement is eventually detected and the nervous system reacts to stabilize the person's body. The process is like this: equilibrium → a perturbation → growing deviation → nervous system response → correction → equilibrium → new perturbation →⋯. Therefore, we may build our model with three parts:

(1) the positive feedback of the mechanical body; (2) the negative feedback of the conscious control; and (3) the noise (perturbation). More detailed descriptions of the three parts are given below.

(1) The mechanics of a human body is often described as an inverted linear pendulum [5,6]. Let $x(t)$ represent the transverse displacement of the gravity center of the pendulum at time t, and suppose $x(t) \ll l$, the length of the pendulum. Then we may write the following differential equation:

$$\dot{x}(t) = \alpha x(t),$$

where $\alpha \approx \sqrt{mgl/2I}$, mg and I are the weight and the moment of inertia of the pendulum, respectively. For example, if $l = 170$ cm, then $\alpha \approx 3 s^{-1}$. It should be noted that for a human body, the consideration of its complicated joints and uneven density may cause variation of α around the value, $\sqrt{mgl/2I}$, or even lead to a complicated function for α.

(2) It is not straightforward to represent how a person's nervous system processes a signal. Here, we consider the whole nervous system as a "black box" so that we can concentrate on the analysis of possible output from it. The output from the black box must lead to negative feedback with a delay in order to balance the human body. The negative feedback may be described by $\beta \tanh[x(t-\tau)]$, where β (<0) is the feedback coefficient, and τ represents the delay. $\tanh(x)$, widely used in neural networks [7], represents a smoothed on switch for feedback when displacement occurs, and it is known as a standard transfer function. For a normal person, τ is estimated to be between 300 and 800 ms [3,5].

(3) All perturbations are represented by white noise, $\gamma\eta(t)$, where γ is the magnitude of the noise, satisfying $\langle \eta(t) \rangle = 0$, and $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$.

Summarizing the above discussions leads to a model for human postural control system:

$$\dot{x}(t) = \alpha x(t) + \beta \tanh[x(t-\tau)] + \gamma\eta(t). \tag{1}$$

Equation (1) describes the dynamic behavior of the individual's center of gravity (COG) in the transverse plane. COP is the projection of COG on the platform where the individual stands when $x \ll l$. Therefore, COG is the same as COP except in different planes.

Now in order to make a comparison between the solution of model (1) and the experimental data, we may apply a numerical simulation to find the solution of system (1) and then use the solution to plot the function $C(\Delta t)$. Figures 1(a–c) show the simulation results of $C(\Delta t)$ for different values of the coefficients. There the slope of $\log_{10} C(\Delta t)$ versus $\log_{10}(\Delta t)$ equals $2H$. Figure 1(a) shows that, similar to the experimental result [1], there exists a larger interval of negative feedback (i.e., $0 < H < 0.5$) when α approaches $|\beta|$ (for example, $\alpha = 1.58$, $\beta = -1.6$). For the curve corresponding to $\alpha = 1.58$, when $\log_{10}\Delta t < 0$, $H \approx 0.7$ (the dashed line shows the average slope in the considered region), and when $\log_{10}\Delta t \in (0.3,1.3)$, $H \approx 0.4$. This may indicate that model (1) correctly describes some properties of human postural control. It is found that the length of the interval for the negative feedback reduces to 0 quickly when α is reduced

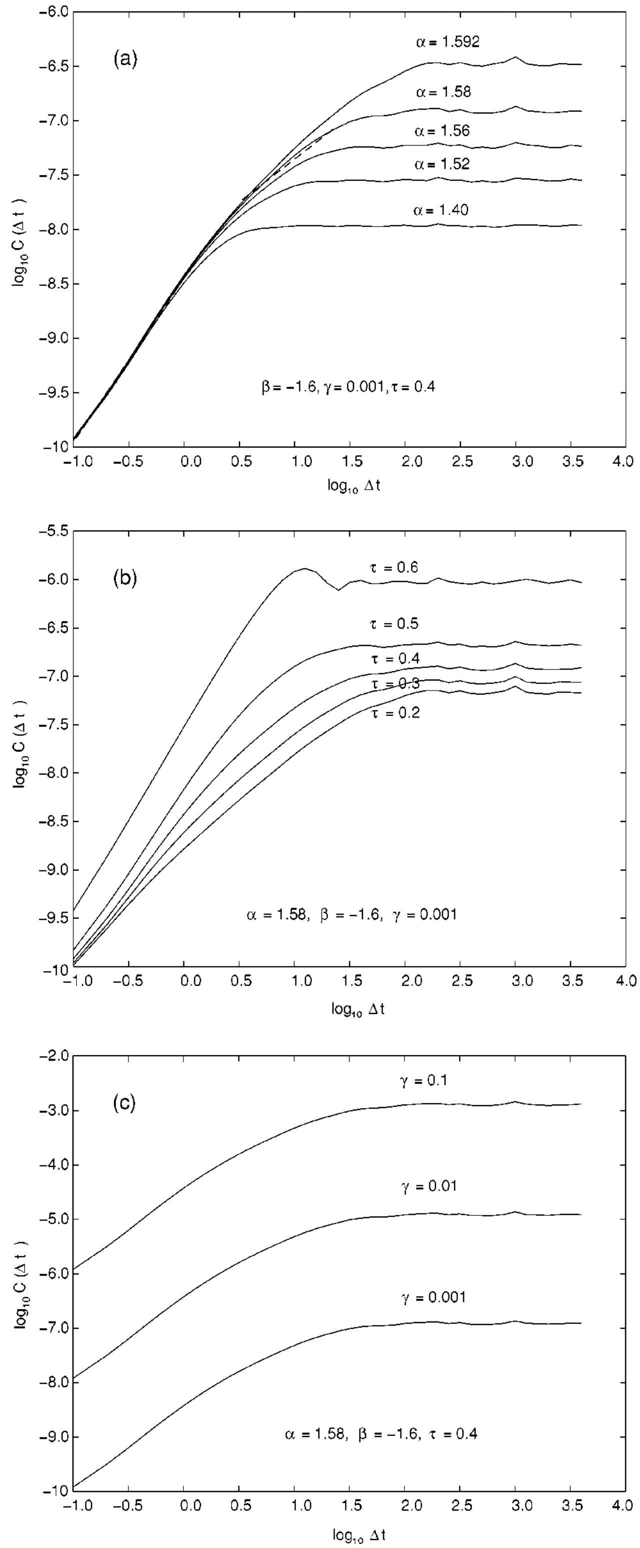

FIG. 1. Numerical simulation of Eq. (1) for function $C(\Delta t)$ with time step 0.01, and initial values $x(t) = 0$ when $t \in (-\tau, 0]$, $\log_{10} C(\Delta t)$ vs $\log_{10}\Delta t$: (a) $\alpha = 1.4$, 1.52, 1.56, 1.58, 1.592, $\beta = -1.6$, $\gamma = 0.001$, $\tau = 0.4$; (b) $\alpha = 1.58$, $\beta = -1.6$, $\gamma = 0.001$, $\tau = 0.2$, 0.3, 0.4, ,0.5, 0.6 (note that the interval of negative feedback tends to disappear when $\tau > 0.5$); (c) $\alpha = 1.58$, $\beta = -1.6$, $\tau = 0.4$, $\gamma = 0.001$, 0.01, 0.1.

from $|\beta|$ (for example, $\alpha=1.40$, $\beta=-1.6$). When α is approximately equal to $|\beta|$, on the other hand, the equilibrium state of the system becomes unstable (i.e., COP leaves the equilibrium point with almost 100% possibility). Figure 1(b) shows $C(\Delta t)$ for different values of τ. It can be observed from this figure that the interval of negative feedback gradually reduces to zero as τ increases. Finally, Fig. 1(c) seems to suggest that the strength of noise does not have much effect on the shape of function $C(\Delta t)$.

It is easy to see from Eq. (1) that the system is more stable for a smaller value of α when β is fixed. For example, when $\beta=-1.6$, the model with $\alpha=1.40$ is more stable than the model with $\alpha=1.58$, because a larger value of $|\beta|/\alpha$ means a larger negative feedback. However, experimental results [1] have shown that the subjects adjust themselves to favor larger values of α (e.g., $\alpha=1.58$) as opposed to smaller ones (e.g., $\alpha=1.4$). Does this mean that people favor instability? To answer the question, we will study a Hopf bifurcation of system (1) in the next section, to answer this based on the stability of Hopf bifurcation.

III. HOPF BIFURCATION

It will be much easier to study Hopf bifurcation of system (1) if the noise term is not present. In this regard, note that the noise given in system (1) is an additive term. According to Ref. [8], the critical points of the bifurcation for the system are independent of the additive noise term. So we may simply ignore the noise term in Eq. (1), setting $\gamma=0$. Alternatively, we could show that $\langle x(t)\rangle \approx x_{\mathrm{det}}(t)$, where $x(t)$ is the solution of Eq. (1) (the notation $\langle\cdots\rangle$ denotes the average over trials) while $x_{\mathrm{det}}(t)$ is the deterministic solution of Eq. (1) when $\gamma=0$. The deterministic solution then represents the average behavior of the stochastic system.

Suppose that one has made a number of trials, and for the ith trial, let the solution of Eq. (1) be $x_i(t)$, which can always be written as $x_i(t)=\langle x(t)\rangle+\epsilon_i(t)$, where the term $\epsilon_i(t)$ is produced by the noise term. Obviously, $\langle\epsilon_i\rangle=\Sigma_i\epsilon_i=0$. Furthermore, notice that

$$\left\langle\frac{dx_i(t)}{dt}\right\rangle=\frac{d}{dt}\langle x(t)\rangle$$

and

$$\begin{aligned}\langle (x_i(t))^n\rangle &= \langle\langle x(t)\rangle^n+n\langle x(t)\rangle^{n-1}\epsilon_i+O(\epsilon_i^2)\rangle\\ &=\langle x(t)\rangle^n+\langle O(\epsilon_i^2)\rangle\end{aligned}$$

Since

$$\tanh[x(t)]=x(t)-\tfrac{1}{3}x^3(t)+\cdots,$$

we obtain

$$\langle\tanh[x_i(t)]\rangle=\tanh[\langle x(t)\rangle]+\langle O(\epsilon_i^2)\rangle.$$

The magnitude of ϵ_i depends on the value of γ and whether or not random resonance appears. If γ is small and no random resonance exists, then ϵ_i must be small too. For the model considered in this paper, the higher order form $O(\epsilon_i^2)$ can be neglected. Therefore, by averaging both sides of Eq. (1) over trials, we finally obtain (for white noise)

$$\dot{x}(t)=\alpha x(t)+\beta\tanh[x(t-\tau)], \tag{2}$$

where, for simplicity, we have dropped the $\langle\rangle$ notation.

A. Conditions for Hopf bifurcation

It is easy to see from Eq. (2) that $x=0$ is an equilibrium point of the system. The linearized equation can be written as

$$\dot{u}(t)=\alpha u(t)+\beta u(t-\tau), \tag{3}$$

which yields the eigenvalue relation,

$$\lambda=\alpha+\beta e^{-\lambda\tau}. \tag{4}$$

In general, λ is complex. Let $\lambda=a+ib$, where a and b take real values. Substituting this expression into Eq. (4) results in

$$a=\alpha+\beta e^{-a\tau}\cos b\tau, \tag{5a}$$

$$b=-\beta e^{-a\tau}\sin b\tau. \tag{5b}$$

Now suppose that system (2) exhibits a Hopf bifurcation which occurs at $a=0$. Setting $a=0$ in Eqs. (5a) and (5b) gives

$$b=\pm\sqrt{\beta^2-\alpha^2} \tag{6}$$

and

$$\alpha+\beta\cos(\tau\sqrt{\beta^2-\alpha^2})=0. \tag{7}$$

If we take τ as the bifurcation parameter, then it follows from Eqs. (5a) and (5b) that

$$\frac{\partial a}{\partial\tau}=\beta e^{-a\tau}\left(-\frac{\partial a}{\partial\tau}\tau-a\right)\cos b\tau-b\beta e^{-a\tau}\sin b\tau \tag{8}$$

and

$$\left.\frac{\partial a}{\partial\tau}\right|_{a=0}=\frac{\beta^2-\alpha^2}{1-\alpha\tau}. \tag{9}$$

It is seen from Eqs. (6) and (9) that when $|\beta|\neq|\alpha|$, we have $b\neq0$ and $[(\partial a)/(\partial\tau)]|_{a=0}\neq0$, which indicates that system (2) has a Hopf bifurcation when α, β, and τ satisfy condition (7), and all other eigenvalues given in Eq. (4) have negative real parts. The latter condition can be satisfied easily. For example, when $\alpha+\beta<0$ and $\tau=0$, λ is real and less than zero. Because $e^{-\lambda\tau}$ is a continuous function of τ, it is expected that there exists an ϵ such that all eigenvalues in Eq. (4) have strictly negative real parts for $\tau\in[0,\epsilon)$. When

$\tau=\epsilon$, Eq. (4) has a pair of purely imaginary eigenvalues and the remaining still have strictly negative real parts. (For more detail, see Ref. [9]).

Note that α or β can also be taken as the bifurcating parameter, and the analysis is similar to the above discussion. In this paper, we will choose τ as the bifurcation parameter since it plays a more important role in the analysis of the dynamical behavior of the system than other parameters do.

B. Stability of Hopf bifurcation

The method of analyzing the stability of Hopf bifurcation for delayed differential equations was introduced by Hassard, Kazarrinoff, and Wan [10], using normal form theory and center manifold theory. The reader is also referred to the recent work by Li, Ruan, and Wei [9].

First, we apply Taylor expansion to Eq. (2) to have a system including terms up to the third order, which is, in general, enough for stability analysis of Hopf bifurcation:

$$\dot{u}(t)=\alpha u(t)+\beta u(t-\tau)-\tfrac{1}{3}\beta u^3(t-\tau)+O(u^4). \quad (10)$$

Let τ_0 be the critical bifurcation value of τ satisfying Eq. (7), and then denote $\mu=\tau-\tau_0$. Further, let $u_t(\theta)=u(t+\theta)$, $\theta\in[-\tau,0)$ and $L_\mu u_t=\alpha u+\beta u(t-\tau)$, where L_μ is a one-parameter family of continuous (bounded) linear operators defined as $L_\mu:C[-\tau,0]\to R$. Then it follows from Eq. (10) that

$$\dot{u}=L_\mu u_t+f(u_t,\mu), \quad (11)$$

where the operator $f(u_t,\mu):C[-\tau,0]\to R$ contains the nonlinear terms, beginning with at least quadratic terms.

With the Riesz representation theorem, one can prove that there exists a function

$$\xi(\theta,\mu):[-\tau,0]\to R,$$

with bounded variation for each component. Moreover, for all $\phi\in C^1[-\tau,0]$,

$$L_\mu\phi=\int_{-\tau}^{0}d\xi(\theta,\mu)\phi(\theta),$$

and in particular,

$$L_\mu u_t=\int_{-\tau}^{0}d\xi(\theta,\mu)u(t+\theta).$$

For our case, by defining

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi}{d\theta}, & \theta\in[-\tau,0)\\[2ex] \displaystyle\int_{-\tau}^{0}d\xi(s,\mu)\phi(s)\equiv L_\mu\phi, & \theta=0,\end{cases}$$

and

$$R\phi=\begin{cases}0, & \theta\in[-\tau,0)\\ f(\phi,\mu), & \theta=0,\end{cases}$$

where

$$\xi(\theta,\mu)=\begin{cases}\beta\delta(\theta+\tau), & \theta\in[-\tau,0),\\ \alpha\delta(\theta), & \theta=0,\end{cases}$$

and noting that $(du_t)/(dt)=(du_t)/(d\theta)$, we can rewrite Eq. (11) as an equation of one variable:

$$\dot{u}_t=A(\mu)u_t+Ru_t. \quad (12)$$

At the critical point for Hopf bifurcation, $A(0)q(\theta)=i\omega_0 q(\theta)$, where $q(\theta)\in[-\tau,0]$ is the eigenvector corresponding to the eigenvalue $\lambda(0)$ of $A(0)$. Here, $\omega_0=b$ given by Eq. (6).

The adjoint operator $A^*(0)$ is defined by

$$A^*(0)\psi(s)=\begin{cases}-\dfrac{d\psi}{ds}, & s\in(0,\tau]\\[2ex] \displaystyle\int_{-\tau}^{0}d\xi(t,0)\psi(-t), & s=0.\end{cases}$$

Assume q^* is the eigenvector corresponding to the eigenvalue $-i\omega_0$ of $A^*(0)$, then we may obtain

$$q(\theta)=e^{i\omega_0\theta},$$

$$q^*(s)=De^{i\omega s}, \quad (13)$$

$$D=\frac{1+\beta\tau_0 e^{i\omega_0\tau_0}}{1+\beta^2\tau^2+2\beta\tau\cos(\omega_0\tau_0)},$$

which, in turn, yields

$$\begin{aligned}1&=\langle q^*,q\rangle\\ &\equiv\overline{q}^*(0)q(0)-\int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{q}^*(\xi-\theta)d\xi(\theta,0)\phi(\xi)d\xi,\end{aligned}$$

and $\langle q^*,\overline{q}\rangle=0$.

For u_t being a solution of Eq. (10) at $\mu=0$, we define

$$z(t)=\langle q^*,u_t\rangle$$

and

$$\begin{aligned}W(t,\theta)&=u_t(\theta)-z(t)q(\theta)-\overline{z(t)}\overline{q}(\theta)\\ &=u_t(\theta)-2\,\mathrm{Re}\{z(t)q(\theta)\}.\end{aligned}$$

Then on the manifold C_0, we can find

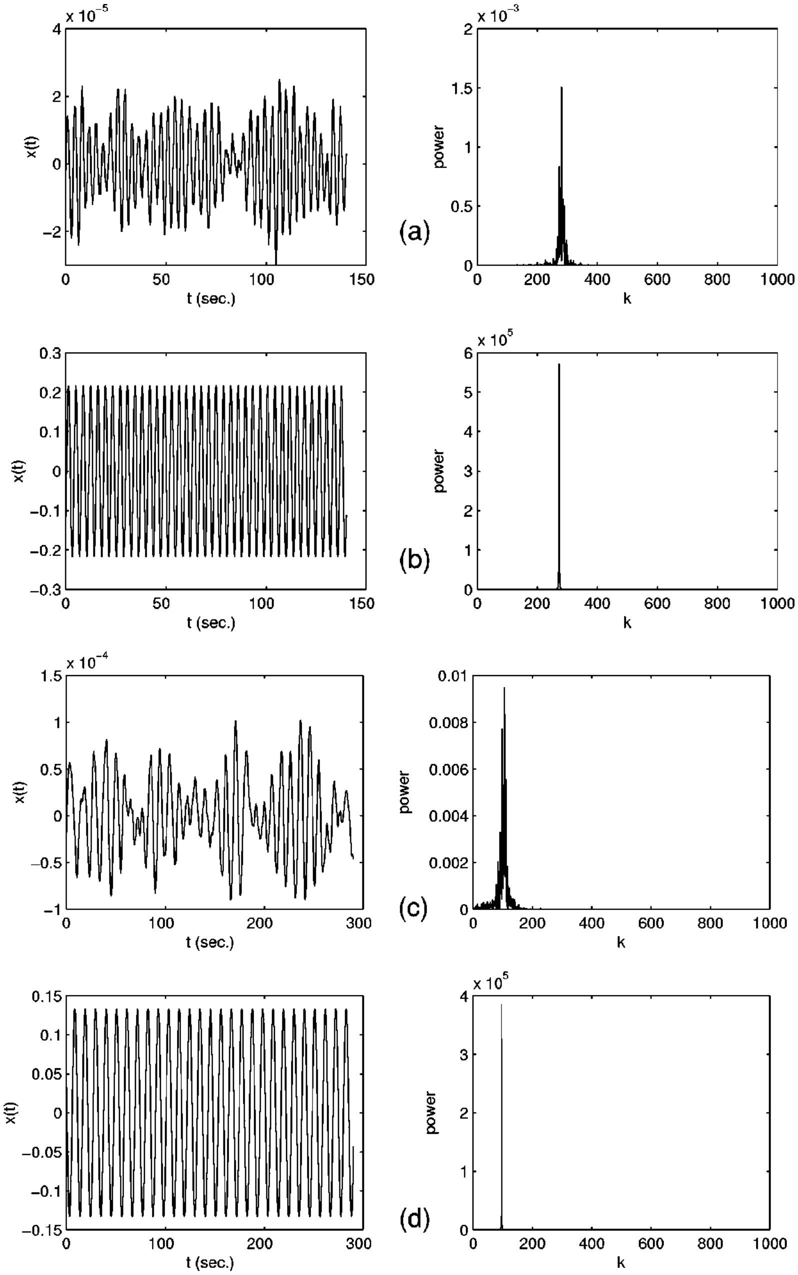

FIG. 2. Time history and power spectrum of $x(t)$ for: (a) $\alpha=1,\beta=-2,\tau=0.6$; (b) the same as (a) except that $\tau=0.62$; (c) $\alpha=1.9,\beta=-2,\tau=0.5$; (d) the same as (c) except that $\tau=0.51$; (e) $\alpha=-1,\beta=-2,\tau=1.2$; and (f) the same as (e) except that $\tau=1.22$.

$$W(t,\theta)=W(z(t),\overline{z}(t),\theta)$$
$$=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots.$$

In fact, z and $\overline{z}$ are local coordinators for C_0 in C in the directions of q^* and $\overline{q}^*$, respectively.

For a solution $u_t\in C_0$, Eq. (11) yields

$$\langle q^*,\dot{u}_t\rangle=\langle q^*,A(0)u_t+Ru_t\rangle,$$

and thus,

$$\dot{z}(t)=i\omega_0 z(t)+\overline{q}^*(0)f_0(u_t)$$
$$=i\omega_0 z(t)+g(z,\overline{z}), \tag{14}$$

where $f_0(u_t)=f(u_t)|_{\theta=0}$, and

$$g(z,\overline{z})=\overline{q}^*(0)f(u_t,0)$$
$$=\overline{q}^*(0)\left[-\frac{\beta}{3}u^3(t-\tau_0)\right]$$
$$=-\frac{\beta}{3}\overline{q}^*(0)[W(z(t),\overline{z}(t),\tau_0)$$
$$+z(t)q(-\tau)+\overline{z}(t)\overline{q}(-\tau)]^3$$
$$=-\frac{\beta}{3}\overline{q}^*(0)\left[W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}\right.$$
$$\left.+\cdots+z(t)q(-\tau)+\overline{z}(t)\overline{q}(-\tau)\right]^3 \tag{15}$$

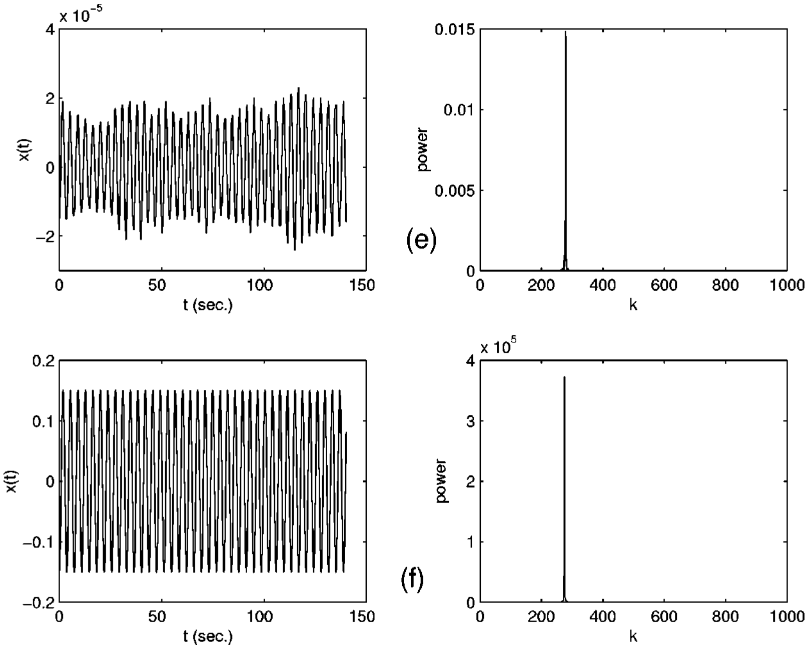

FIG. 2. (*Continued*).

$$=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots. \tag{16}$$

Now by comparing the coefficients of $z^2, z\bar{z}$, $\bar{z}^2$ and $z^2\bar{z}$ in Eq. (15) with those in Eq. (16), we obtain

$$g_{20}=g_{11}=g_{02}=0,$$

$$g_{21}=-2\beta\bar{q}^*(0)q^2(-\tau_0)\bar{q}(-\tau_0)=-2\beta\frac{e^{-i\omega_0\tau_0}}{1+\beta\tau_0e^{-i\omega_0\tau_0}}.$$

Finally, the stability of Hopf bifurcation can be determined by [10]

$$\beta_2=2\,\mathrm{Re}\{c_1(0)\},$$

and the direction of Hopf bifurcation is determined by

$$\mu_2=-\frac{\mathrm{Re}\{c_1(0)\}}{\left.\frac{\partial a}{\partial\tau}\right|_{a=0}},$$

where

$$c_1(0)=\frac{i}{2\omega_0}(g_{20}g_{11}-2|g_{11}|^2-\tfrac{1}{3}|g_{02}|^2)+\frac{g_{21}}{2}.$$

When $\beta_2<0(>0)$, the Hopf bifurcation is stable (unstable); when $\mu_2>0(<0)$, the Hopf bifurcation is supercritical (subcritical). For our model, the stability conditions are given by

$$\beta_2=\frac{2(\alpha-\beta^2\tau_0)}{1-2\alpha\tau_0+\beta^2\tau_0^2}, \tag{17}$$

$$\mu_2=-\frac{(\alpha-\beta^2\tau_0)(1-\alpha\tau_0)}{(1-2\alpha\tau_0+\beta^2\tau_0^2)(\beta^2-\alpha^2)}. \tag{18}$$

Equation (17) indicates that Hopf bifurcation of system (2) is always stable if $\alpha\leqslant 0$.

C. Numerical simulation of Hopf bifurcation

To verify that the results obtained from the bifurcation analysis of system (2) are true for the original system (1), we use numerical simulation to find the solution of system (1) in the vicinity of the Hopf bifurcation critical point defined by Eq. (7). Figure 2 shows the time history and power spectrum of $x(t)$ for different values of α, β, and τ, with a fixed noise strength $\gamma=0.0001$. The frequency f can be calculated using $f=(k-1)/1000$. The step size and initial conditions are the

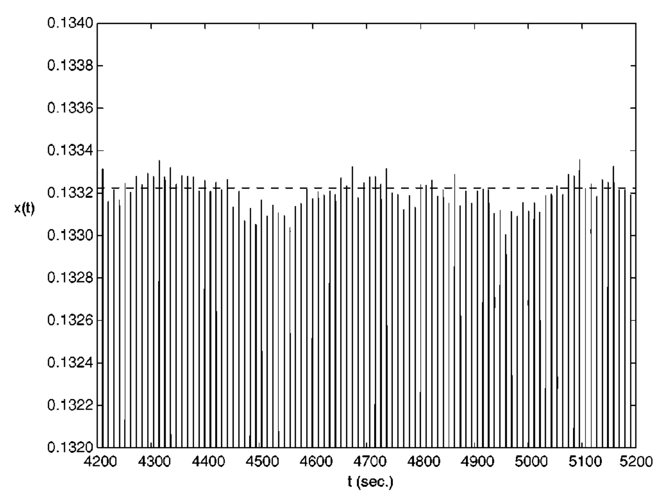

FIG. 3. Windowed time history of $x(t)$ taken from Fig. 2(d) for $0.132\leqslant x(t)\leqslant 0.134$.

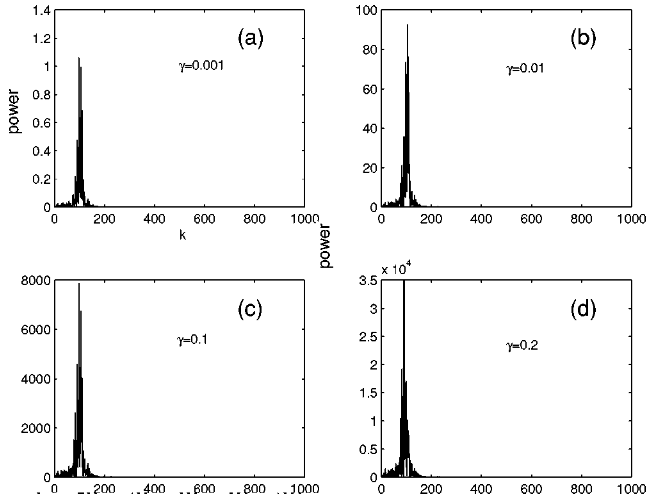

FIG. 4. Power spectrum of $x(t)$ at different noise strength for $\alpha=1.9,\beta=-2,\tau=0.50$: (a) $\gamma=0.001$; (b) $\gamma=0.01$; (c) $\gamma=0.1$; and (d) $\gamma=0.2$.

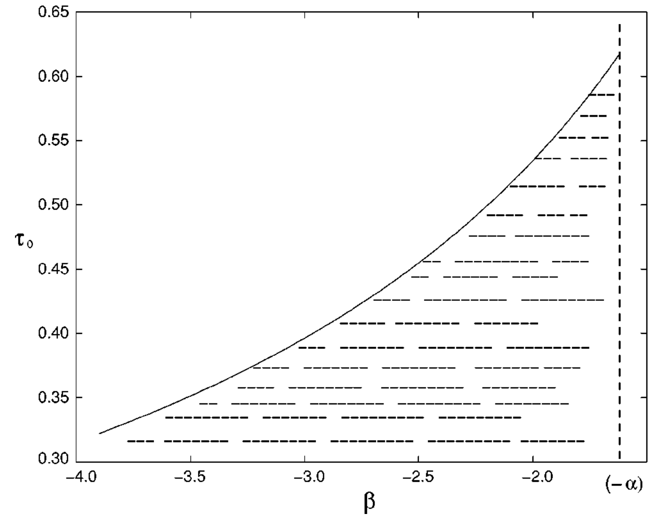

FIG. 5. The critical values of τ_0 under Hopf bifurcation.

same as that given in Fig. 1. When $\alpha=1,\beta=-2$, Eq. (7) gives the critical value $\tau_0=\pi/3\sqrt{3}\approx 0.605$, and then it follows from Eqs. (17) and (18) that $\beta_2<0$ and $\mu_2>0$. Therefore, the Hopf bifurcation, when $\alpha=1$, and $\beta=-2$, is stable and supercritical. When τ is varied from $\tau<\tau_0$ to $\tau>\tau_0$, the system will experience a Hopf bifurcation as shown in Fig. 2. It is observed from the power spectrum of $x(t)$ that when τ approaches τ_0, there exists a relatively large frequency spectrum near $f_0=\omega_0/2\pi$. This spectrum is not only different from that of the deterministic case where there is no oscillation, but also different from that of white noise case where there is no dominant frequency. It is noted that, before τ reaches τ_0, noise always destabilize the system (deviating from the equilibrium), but the system eventually converges to the equilibrium. The frequency of the oscillation is close to that evaluated at the critical point of Hopf bifurcation. However, by comparing the magnitudes of the frequency spectrum of $x(t)$ for $\tau\geqslant\tau_0$ with that for $\tau<\tau_0$, we may find that the behavior of the system is much more disordered when $\tau<\tau_0$. Therefore, the system indeed exhibits Hopf bifurcation at $\tau=\tau_0$. The simulation results confirm the analytic predictions obtained in previous subsections. Moreover, it is noted that the amplitudes of $x(t)$ become relatively large when $\tau>\tau_0$, which agrees with the observation of quiet standing.

The result that we proved at the beginning of Sec. III, i.e., the deterministic solution of system (2) represents the average behavior of the stochastic system (1) if the stochastic term is additive, is verified numerically as shown in Fig. 3. This figure gives a windowed time history of $x(t)$ depicted in Fig. 2(d). The dashed line represents the amplitude of $x_{\text{det}}(t)$. It is clear that the amplitude of $x(t)$ stochastically oscillates around that of $x_{\text{det}}(t)$.

Furthermore, it is interesting to find, from the numerical investigations, how the noise effects Hopf bifurcation. As shown in Fig. 4, the system oscillates more periodically in terms of power when the noise strength is increasing. Noise helps the system enter into a stochastically resonant state if the system is near Hopf bifurcation, or alternatively the oscillations are coopting energy from the noise. However, the bifurcation point is not affected even for rather strong noise strength. Thus, we may conclude that the deterministic solution of system (2) indeed represents the average behavior of system (1) for small γ.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have proposed a physiological model for describing human postural control. The model is a delayed stochastic differential equation, which is simple enough for us to analytically study Hopf bifurcation and obtain the explicit stability conditions. Numerical simulations show that the simple model reproduces the behavior of the two-point autocorrelation function $C(\Delta t)$, which is obtained from experimental data. The parameter values for such a function $C(\Delta t)$ have been carefully analyzed.

The results shown in Fig. 1(a) indicate that a larger interval of negative feedback only exists when $0<|\beta|-\alpha\ll 1$. The critical value of the delay determined from Eq. (7) is found as $\tau_0=\arccos(-\alpha/\beta)/\sqrt{\beta^2-\alpha^2}$, which suggests that given a value of α, τ_0 increases monotonically as $-\beta$ decreases to the value of α (see Fig. 5 which shows the situation when $\alpha=1.6$). On the other hand, substituting the above expression of τ_0 into Eqs. (17) and (18) yields $\beta_2<0$ and $\mu_2>0$, respectively. Therefore, the equilibrium point located in the shadowed area (see Fig. 5) is entirely stable, and the Hopf bifurcation shown in Fig. 5 is stable too. However, in reality, one hopes to avoid Hopf bifurcation because it means large amplitude oscillation as shown in Fig. 2. This suggests that the critical value of τ_0 is better to be as large as possible, so that a person can have enough time to react or adjust. In other words, the form of function $C(\Delta t)$ obtained from experimental data [1] corresponds to the case of our model when $0<|\beta|-\alpha\ll 1$, which implies that maximum delay τ_0

helps people to avoid oscillations (Hopf bifurcation). The above discussion seems to answer the question raised in the introduction, leading to the following conclusion: Human postural control seems to be an optimal process to avoid possible oscillations (Hopf bifurcation). This capability may be learned and it may be affected by disease. It may also be possible to diagnose some disease by studying the COP of the subject [11].

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